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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF POTENTIAL THEORY AND THE THEORY OF ELASTICITY IN THE VICINITY OF CONICAL POINTS ON THE BOUNDARY†

P. I. PERLIN

Moscow

One-dimensional integral equations of the second kind are constructed in order to determine the asymptotic behaviour of the solutions of the boundary value problems of potential theory and the theory of elasticity in the vicinity of conical points on the boundary surface. An algorithm for solving the problems is described and computational results for model examples are presented.

As A RESULT of studying the solutions of the boundary value problems for elliptic equations in the vicinity of conical points on the boundary surface it has been shown [1] that, in any domain of this type, the solution can be represented as the sum of an infinitely differentiable function and an asymptotic series, each term of which is a solution of the homogeneous boundary value problem for an infinite cone formed by the half-lines tangent at the conical point. The solutions in question (eigenfunctions) are determined only by the local structure of the conical surface and the type of boundary conditions. Naturally, the coefficients multiplying the solutions depend on the general configuration of the domain and the values of the boundary conditions. It is obvious that the

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construction of such solutions is easier than the solution of the whole boundary value problem. Thus, by determining the solutions in advance and including them in some form in the computational schemes, one can make it much easier to solve the original boundary value problem.

We note that in problems with physical content one has to discard the eigenfunctions that result in the energy being unbounded. Of special interest in the mathematical context are those eigenfunctions whose derivatives that appear in the computational schemes become infinite. The coefficient multiplying those eigenfunctions, whose first-order derivatives are infinite, play an important role in a number of applications (for example in brittle fracture mechanics).

In what follows we will restrict ourselves to problems for Laplace's equation and the theory of elasticity. When determining the eigenfunctions for cones, it is customary to introduce the spherical coordinates r, φ, and θ and to employ the method of partial separation of variables. We will define the surface of the cone by the following equations and inequalities:

$$0 \leqslant r < \infty, \; 0 \leqslant \varphi \leqslant 2\pi, \; \theta = \theta \, (\varphi)$$

For Laplace's equation, starting from the representation of the solution in the form

$$u \, (r, \; \theta, \; \varphi) = r^{\lambda} U \, (\theta, \; \varphi) \tag{1}$$

we obtain a two-dimensional boundary value problem for the eigenvalues for the function $U(\theta, \varphi)$, which satisfies the equation

$$\lambda \, (\lambda + 1) \, U + \frac{1}{\sin \theta} \, \frac{\partial}{\partial \theta} \left(\sin \theta \, \frac{\partial U}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \, \frac{\partial^2 U}{\partial \varphi^2} = 0 \tag{2}$$

with homogeneous boundary conditions.

For the Lamé equations one can start from the representations

$$u_r \, (r, \; \theta, \; \varphi) = r^{\lambda} U_r \, (\theta, \; \varphi), \; u_{\theta} = r^{\lambda} U_{\theta} \, (\theta, \; \varphi), \; u_{\varphi} = r^{\lambda} U_{\varphi} \, (\theta, \; \varphi) \tag{3}$$

of the displacement components in spherical coordinates.

Note that in the sequel we shall use the projections of the displacement vector onto the Cartesian coordinate axes, which have an analogous structure.

The solutions of a number of specific problems [2–5] have been obtained on the basis of the approach described.

Below we propose another method, which is based on the formalism of potential theory and reduces the problem to a one-dimensional integral equation (IE) on a contour lying on the conical surface.

As we know, the basic spatial problems of potential theory and the theory of elasticity can be reduced to two-dimensional IEs (regular and singular ones) on the boundary surface. Those equations can be obtained by representing the functions to be determined in the form of single- or double-layer potentials (the indirect approach) as well as on the basis of the Green and Betti formulae (the direct approach; see e.g. [6, 7]).

We will begin with the Neumann problem for Laplace's equation. According to (1), the representation

$$u \, [r, \; \theta \, (\varphi), \; \varphi] = r^{\lambda} U^* \, (\varphi) \tag{4}$$

holds on the boundary surface of the cone. We now consider the integral equation

$$u \, (q) - \frac{1}{2\pi} \int_{S} u \, (q) \, \frac{d}{dn_{q_1}} \left[\frac{1}{R \, (q, q_1)} \right] dS_{q_1} = 0, \quad R = |q - q_1| \tag{5}$$

which can be obtained from Green's formula (the direct approach). Here S is the surface of the cone, and q and q_1 are points on the cone with coordinates r, φ, $\theta(\varphi)$ and r_1, φ_1, $\theta(\varphi)_1$ $(r \neq 0)$.

According to (4), we can transform Eq. (5) to the form

$$r^\lambda U^*(\varphi) - \frac{1}{2\pi} \int_0^{2\pi} U^*(\varphi_1) \cos \theta(\varphi_1) \left\{ \int_0^\infty r_1^{\lambda+1} \frac{d}{dn_{q_1}} \left[\frac{1}{R(q, q_1)} \right] dr_1 \right\} d\varphi_1 = 0 \tag{6}$$

This equation must be satisfied on the whole surface of the cone, but it suffices to require that it be satisfied at a single point on each generatrix, i.e., in fact, on any contour lying on the surface with the conical point inside. The latter assertion follows from the fact that the (radial) structure of the solution of the IE is known and only the function $U^*(\varphi)$ of one argument is to be determined. There is no need to present the expanded form of the resulting one-dimensional IE. We remark that, according to the traditional approach, the boundary value problem is to be solved in a fixed domain (the part of a sphere inside the cone) and a sufficiently arbitrary choice of the contour for the IE is admitted, which can be used to simplify the numerical realization.

From the structure of the integrand of the inner integral it follows that, for the integral to converge, one has to introduce some restrictions on the admissible values of λ $(0 < \lambda < 1)$. The restrictions are not important, since the eigenfunctions have unbounded square-integrable first-order derivatives (precisely the most interesting ones).

Therefore the problem has been reduced to determining those values of λ for which the homogeneous equation (6) has a non-trivial solution. Using one of the methods for the numerical solution of IEs, we can determine the required values of the parameter by selection and interpolation. For example, if the mechanical cubature method is used, one has to divide the whole surface of the cone into elementary segments by a bundle of rays, and, having set $U^*(\varphi)$ to be constant within each of those segments, to integrate by means of any cubature formula, successively placing the points q in the middle (relative to the angle) of each segment at an arbitrary distance from the tip. As a result, one can obtain a system of algebraic equations. We compute the determinant, and, if it turns out to be non-zero (to within the prescribed accuracy), we can pass to another value of the parameter. Having determined the parameter, we can pass to the construction of the eigenfunction of the IE. To do this, it is necessary to express all the values of $U^*(\varphi)$ (at the discrete points q_j) in terms of one of them. We complete the algorithm by constructing the eigenfunction of the boundary value problem. We use Green's formula (taking into account that the derivative vanishes)

$$u(r, \theta, \varphi) = r^\lambda U(\theta, \varphi) = \frac{1}{2\pi} \int_S r_1^\lambda U^*(\varphi_1) \frac{d}{dn_{q_1}} \left[\frac{1}{R(q, q_1)} \right] dS_{q_1} \tag{7}$$

Let us consider the special form of the solution of the Neumann problem using the IE obtained on the basis of the indirect approach. Above, the solution of the IE has been a restriction of the solution of a boundary value problem. Thus the order of the asymptotic forms of the boundary value problem has been the same as that of the IE, and, according to (7), the eigenfuntion of the IE has led to the eigenfunction for the boundary value problem. For those IEs that can be obtained on the basis of a simple layer potential, it was established [8] that the asymptotic expressions can be determined as those for the solution of the boundary value problem for the given cone as well as the complementary one. Therefore it turns out that, having determined the eigenfunction of the IE, one has to verify the boundary condition for the eigenfunction of the boundary value problem.

Similar complications also arise for the solution of the Dirichlet problem based on the potential of a double layer.

The approach presented above can be extended to the problems of elasticity theory. We shall consider the construction of vector-valued eigenfunctions for the second fundamental problem. We will begin with the singular IE resulting from the Betti formula (the direct approach). We have the equation

$$\mathbf{u}(q) + \int_S \mathbf{\Gamma}_2(q, q_1) \mathbf{u}(q_1) \, dS_{q_1} = 0 \tag{8}$$

Here $\mathbf{\Gamma}_2(q, q_1)$ is the kernel of the double layer potential and $\mathbf{u}(q)$ is a vector-valued function representing the displacements on the surface of the cone. In accordance with (3), we introduce the following representations for the components of the displacements in Cartesian coordinates:

$$u = r^\lambda U^*(\varphi), \quad v = r^\lambda V^*(\varphi), \quad w = r^\lambda W^*(\varphi) \tag{9}$$

Compared with harmonic problems complications arise connected with the fact that one has to compute two-dimensional singular integrals. In order to compute these integrals, it has been proposed to use the so-called regular representations [9], which transform the integrals into non-singular ones. However, since the boundary surface is not closed, one has to use a modification of the regular representations [10]. Let S_1 be the part of the surface of the cone over which the integration is in fact carried out. We will introduce a surface S_2 supplementing S_1 so that together they form a closed surface. Then the regular representation can be written in the form

$$\int_{S_1} \mathbf{\Gamma}_2(q, q_1) \mathbf{u}(q_1) \, dS_{q_1} = -\mathbf{u}(q) + \int_{S_1} \mathbf{\Gamma}_2(q, q_1) [\mathbf{u}(q_1) - \mathbf{u}(q)] \, dS_{q_1} -$$
$$- \mathbf{u}(q) \int_{S_2} \mathbf{\Gamma}_2(q, q_1) \, dS_{q_1} \tag{10}$$

In order to demonstrate the efficiency of the proposed method, computations have been carried out for problems whose solutions can be obtained in the traditional way. Thus, as a result of computations performed for the Dirichlet problem in the case of a circular cone ($\theta = \theta_0$) it has been shown [3] that an axially symmetric solution corresponds to λ between 0 and 1. This fact has been assumed to simplify the computations carried out in accordance with the IE obtained on the basis of the potantial of a double layer. Thus the left-hand side of the integral equation has been computed only as a single point on the surface (lying at unit distance from the vertex in the direction $\varphi = 0$ and $\theta = \theta_0$). The IE takes the form

$$1 - \frac{\cos \theta_0}{2\pi} \int_0^{2\pi} \sin^2 \frac{\varphi_1}{2} I(\varphi_1) \, d\varphi_1 = 0, \quad I(\varphi_1) = \int_0^\infty \left[1 + r_1^2 - 2r_1 \cos^2 \frac{\varphi_1}{2} \right]^{-3/2} r_1^{\lambda+1} \, dr_1 \tag{11}$$

The fact that the inner integral is the Mellin transform of a function, which can be computed in terms of special functions [11], has been used to estimate the accuracy of the valuation of that integral. The following identity holds:

$$\int_0^\infty t^{s-1} (1 + 2t \cos \psi + t^2)^{-\nu} \, dt = 2^{\nu-1/2} (\sin \psi)^{\nu-1/2} \Gamma\left(\frac{1}{2} + \nu\right) \frac{\Gamma(s) \Gamma(2\nu - s)}{\Gamma(2\nu)} P_{s-\nu-1/2}^{1/2-\nu}(\cos \psi) \tag{12}$$

For $\nu = 3/2$, $s = \lambda + 1$, and $\psi = \arccos(1/2\cos^2 \varphi_1)$, the integral on the left-hand side is identical with $I(\varphi_1)$. For these values the right-hand side can be simplified and takes the form

$$\sin \psi \, \Gamma(\lambda - 2) \, \Gamma(1 - \lambda) \, P_\lambda^{-1}(\cos \psi)$$

The calculations were carried out for $\cos\psi = 0$ and $\lambda = 0.5$. According to the tables [12], the value of the right-hand side of (12) turns out to be 1.8541, while the value of the left-hand side (obtained by numerical integration) is 1.8526.

The values (Δ) of the left-hand side of Eq. (11), which are listed below for $\theta_0 = 3/4\pi$, have been obtained as a result of computations carried out for various λs:

λ	0.8	0.7	0.6	0.5	0.4	0.3
Δ	-1.12	-0.53	-0.23	-0.05	-0.07	0.15

The value $\lambda = 0.46$ (as compared to the value 0.4631 in [3]) that corresponds to $\Delta = 0$ has been obtained by interpolation.

The Neumann problem has also been considered. It follows from known results [3] that, in this case the value $\lambda > 1$ corresponds to the axially symmetric solution. This value cannot be determined by the proposed method. The solution $U^*(\varphi) = \cos\varphi$ lies within the required limits. This fact has been assumed in the computations. Thus it has been required that the IE be satisfied at one point, as before. Equation (6) takes the form

$$1 + \frac{1}{2^{3/2}\pi} \int_0^{2\pi} \cos\varphi_1 \sin^2 \frac{\varphi_1}{2} I(\varphi_1)\, d\varphi_1 = 0 \tag{13}$$

The values (Δ) of the left-hand side of (13) listed before have been obtained by computations for various λs:

λ	0.8	0.83	0.84	0.85	0.86	0.87	0.88
Δ	0.493	0.260	0.200	0.127	0.065	-0.050	-0.15

The value $\lambda = 0.8653$ (as compared to 0.8564 in [3]) has been obtained by interpolation.

The algorithm presented can be extended to problems involving a boundary surface that consists of pieces of smooth surfaces with a common point (i.e. a singularity of the type of the vertex of a polyhedral cone). We can start from the same representations for the functions to be determined and the same IEs. We realize the discretization for each piece of the surface separately. Then the problem of constructing discrete analogues of the IE at the points on the edges disappears automatically.

The Dirichlet problem for a trihedral cone formed by the quadrants (the large angle) has been considered. Since the solution of the Dirichlet problem for a circular cone, with the asymptotic form admitted by the method, is axially symmetric, it has been assumed that in the problem in question the edges of the cones as well as the bisectrices of the quadrants are lines of symmetry, and so all the computations have been carried out for points belonging to half a quadrant. Those points have been placed at unit distance from the vertex, providing a uniform angular partition of the half-quadrant into N parts. For $N = 3$, 4, 8, the values obtained for λ are 0.4445, 0.4459, and 0.4510 (the previous value obtained in [4] was 0.455). We remark that the upper and lower estimates $0.433 < \lambda < 464$ have been established for the problem in question [13].

For $N = 4$ and $N = 8$, the eigenfunction of the IE has also been determined. Figure 1 shows a graph of the eigenfunction depending on the angular coordinate of the corresponding point with respect to the bisectrix of the quadrant (the eigenfunction is normalized to be equal to one at a point near the bisectrix).

In Fig. 1 the solid line represents the function for $N = 4$ and the broken line for $N = 8$.
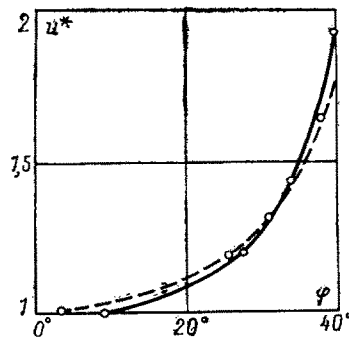

FIG. 1.

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